Stat 155 Lecture 11 Notes

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1 Two-Player and Multiple-Player General-Sum Games

1.1 More about two-player general-sum games

1.1.1 Cheetahs and gazelles

Here is another example of a two-player general-sum game.

Example 1.1. Two cheetahs are chasing a pair of antelopes, one large and one small. Each cheetah has two possible strategies: chase the large antelope (L) or chase the small antelope (S). The cheetahs will catch any antelope they choose, but if they choose the same one, they must share the spoils. Otherwise, the catch is unshared. The large antelope is worth ℓ, and the small one is worth s.

The payoff bimatrix for this game is

	large	small
large	$(\ell/2, \ell/2)$	(ℓ, s)
small	(s, ℓ)	$(s/2, s/2)$

If $\ell \geq 2s$, then large is a dominant strategy. If $\ell \leq 2s$, then the pure Nash equilibria are $(\text{large}, \text{small})$ and $(\text{small}, \text{large})$. What about a mixed Nash equilibrium? If Cheetah 1 plays $\mathbb{P}(\text{large}) = x$, then Cheetah 2's payoffs are

$$\text{large} \quad L(x) = \frac{\ell}{2}x + \ell(1-x),$$

$$\text{small} \quad S(x) = sx + \frac{s}{2}(1-x).$$

Equilibrium is when these are equal:

$$x^* = \frac{2\ell - s}{\ell + s}.$$

For example, if $\ell = 8$ and $s = 6$, then $x^* = 5/7$.

Think of x^* as the proportion of a population that would greedily pursue the large gazelle. For a randomly chosen pair of cheetahs, if $x > x^*$, $S(x) > L(x)$, and non-greedy cheetahs will do better (and vice versa). Evolution pushes the proportion to x^*; this is the evolutionarily stable strategy.

1.1.2 Comparing two-player zero-sum and general-sum games

How do two player general-sum games differ from the zero-sum case?

- Zero-sum games
  - A pair of safety strategies is a Nash equilibrium (minimax theorem)
  - There is always a Nash equilibrium.
  - If there are multiple Nash equilibria, they form a convex set, and the expected payoff is identical within that set.
  - Any two Nash equilibria give the same payoff.
  - If each player has an equalizing mixed strategy (that is, $x^\top A = V\mathbf{1}^\top$ and $Ay = V\mathbf{1}$), then this pair of strategies is a Nash equilibrium (from the principle of indifference).
- General-sum games
  - A pair of safety strategies might be unstable. (opponent aims to maximize their payoff, not minimize mine).
  - There is always a Nash equilibrium (Nash's theorem).
  - There can be multiple Nash equilibria with different payoff vectors.
  - If each player has an equalizing mixed strategy for their opponent's payoff matrix (that is, $x^\top B = V_2\mathbf{1}^\top$ and $Ay = V_1\mathbf{1}$), then this pair of strategies is a Nash equilibrium.

1.2 Multiplayer general-sum games

A k-person general-sum game is specified by k utility functions $U_j : S_1 \times S_2 \times \cdots \times S_k \to \mathbb{R}$. Player j can choose strategies $s_j \in S_j$. Simultaneously, each player chooses a strategy. Player j receives the payoff $u_j(s_1, \ldots, s_k)$. In the case where $k = 2$, we have the familiar $u_1(i, j) = a_{i,j}$ and $u_2(i, j) = b_{i,j}$.

For $s = (s_1, \ldots, s_k)$, we denote s_{-i} as the strategies without the ith one:

$$s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_k).$$

We then write (s_i, s_{-i}) as the full vector.

Definition 1.1. A vector $(s_1^*, \ldots, s_k^*) \in S_1 \times \cdots \times S_k$ is a *pure Nash equilibrium* for utility functions $u_1, \ldots, u_k$ if for each player $j \in \{1, \ldots, k\}$,

$$\max_{s_j \in S_j} u_j(s_j, s_{-j}^*) = u_j(s_j^*, s_{-j}^*).$$

If the players play these s_j^*, nobody has an incentive to unilaterally deviate; each player's strategy is a best response to the other players' strategies.

Definition 1.2. A sequence $(x_1^*, \ldots, x_k^*) \in \Delta_{S_1} \times \cdots \times \Delta_{S_k}$ is a *Nash equilibrium* (also called a *strategy profile*) for utility functions $u_1, \ldots, u_k$ if, for each player $j \in \{1, \ldots, k\}$,

$$\max_{x_j \in \Delta_{S_j}} u_j(x_j, x_{-j}^*) = u_j(x_j^*, x_{-j}^*).$$

Here, we define

$$\begin{aligned} u_j(x^*) &= E_{s_1 \sim x, \ldots, s_k \sim x_k} u_j(s_1, \ldots, s_k) \\ &= \sum_{s_1 \in S_1, \ldots, s_k \in S_k} x_1(s_1) \cdots x_k(s_k) u_j(s_1, \ldots, s_k). \end{aligned}$$

If the players play these mixed strategies x_j^*, nobody has an incentive to unilaterally deviate; each player's mixed strategy is a best response to the other players' mixed strategies.

Lemma 1.1. *Consider a k-player game where x_i is the mixed strategy of player i. For each i, let $T_i = \{s : x_i(s) > 0\}$. Then $(x_1, \ldots, x_k)$ is a Nash equilibrium if and only if for each i, there is a constant c_i such that*

1. *For all $s_i \in T_i$, $u_i(s_i, x_{-i}) = c_i$.*
2. *For all $s_i \notin T_i$, $u_i(s_i, x_{-i}) \leq c_i$.*

Example 1.2. Three firms will either pollute a lake in the following year or purify it. They pay 1 unit to purify, but it is free to pollute. If two or more pollute, then the water in the lake is useless, and each firm must pay 3 units to obtain the water that they need from elsewhere. If at most one firm pollutes, then the water is usable, and the firms incur no further costs.

If firm 3 purifies, the cost trimatrix (cost $= -$ payoff) is

	purify	pollute
purify	$(1,1,1)$	$(1,0,1)$
pollute	$(0,1,1)$	$(3,3,4)$

If firm 3 pollutes, the cost trimatrix is

	purify	pollute
purify	$(1, 1, 0)$	$(4, 3, 3)$
pollute	$(3, 4, 3)$	$(3, 3, 3)$

Three of the pure Nash equilibria are (purify, purify, pollute), (purify, pollute, purify), and (pollute, purify, purify). There is also the Nash equilibrium of (pollute, pollute, pollute), which is referred to as the "tragedy of the commons."

Let $x_i = (p_i, 1-p_i)$ (that is, i purifies with probability p_i). It follows from the previous lemma that these strategies are a Nash equilibrium with $0 < p_i < 1$ if and only if

$$u_i(\text{purify}, x_{-i}) = u_i(\text{pollute}, x_{-i}).$$

So if $0 < p_1 < 1$, then

$$\begin{aligned} p_2p_3 + p_2(1-p_3) + p_3(1-p_2) + 4(1-p_2)(1-p_3) \\ = 3p_2(1-p_3) + 3p_3(1-p_2) + 3(1-p_2)(1-p_3), \end{aligned}$$

or, equivalently,

$$1 = 3(p_2 + p_3 - 2p_2p_3).$$

Similarly, we get

$$1 = 3(p_1 + p_3 - 2p_1p_3),$$

$$1 = 3(p_1 + p_2 - 2p_1p_2).$$

Solving gives us two symmetric Nash equilibria:

$$p_1 = p_2 = p_3 = \frac{3 \pm \sqrt{3}}{6}.$$